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New features of the atomic Wehrl entropy and its density in multi-quanta two-level system

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Abstract

This contribution describes new features of the atomic Wehrl entropy in a multi-quanta two-level system in the presence of the Kerr medium. A definition of the atomic Wehrl entropy is presented for this system, based on the atomic Q-function as information measurement about the atomic phase space. The influence of the nonlinear interaction of the Kerr medium, Stark shift, one and two-photon processes and the detuning parameters on the properties of the atomic Wehrl entropy is examined. The atomic Wehrl entropy gives equivalent results to the von Neumann entropy.

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1. Introduction

Different definitions for entropy have been introduced. The most famous one is the von Neumann entropy [1, 2]. The Wehrl entropy was introduced as a classical information entropic approach to deal with quantum fields [2, 3]. The Wehrl entropy of Bloch coherent state, which is called the atomic Wehrl entropy, has been introduced [4].

The von Neumann entropy, which is used to measure the purity of the quantum state and entanglement between the atom and the field, has been discussed extensively (see, e.g., [5–8]). In order to measure some phenomena on phase space such as purity, entropy, phase locking, phase properties, etc, we have to use some new measurements different from the von Neumann entropy such as the Wehrl entropy [9–12]. These phenomena are very important in the field of quantum information and computation [13], and have been discussed in different initial states of the field [11, 12].

Wehrl's entropy of squeezed states was first calculated in the context of entropic uncertainty relations [3]. The problem as to whether the Wehrl entropy can provide a reasonable classification of states with respect to their nonclassical behaviour has been investigated [14]. To check such a possibility, the explicit values of the Wehrl entropies for various quantum states of light were calculated. Also the usefulness of this concept as a compact yet very informative measure describing properly the time evolution of many quantum systems was clearly demonstrated by analysing the dynamics of two simple but nontrivial quantum optical examples: the formation of finite superpositions of coherent states in a Kerr-like medium [9], and the non-unitary evolution of the one-mode electromagnetic field in the Jaynes–Cummings model [10]. In both cases the Wehrl entropy gives a very transparent description of the relevant dynamics.

The aim of this paper is to investigate the atomic Wehrl entropy of a two-level atom as a measure of atomic phase space uncertainty. We examine the influence of a nonlinear medium, detuning parameter and Stark shift on the properties of atomic Wehrl density in both the one- and two-photon processes. The structure of this paper is as follows. In section 2, we present a general model for the interaction between a two-level atom and a quantized electromagnetic field. The definition of the atomic Wehrl entropy and atomic Wehrl density based on the atomic Q-function definition is introduced in section 3. Finally the numerical results and discussion are presented in section 4.

2. The model

In this section, we briefly review a general model for the interaction between a two-level atom and a quantized electromagnetic field. In the rotating wave approximation, the total Hamiltonian can be written as [1]

$$\mathcal{H} = \hbar\omega_F(\hat{a}^\dagger\hat{a} + k\hat{\sigma}_z) + \hbar\Delta_1\hat{\sigma}_z + \hbar\hat{a}^\dagger\hat{a}(s_1|e\rangle\langle e| + s_2|g\rangle\langle g|)\delta(k-2) + \varkappa\hat{a}^{\dagger 2}\hat{a}^2 + \hbar\lambda[\hat{a}^k\hat{\sigma}_+ + \hat{a}^{\dagger k}\hat{\sigma}_-] \quad (1)$$

where ω_F is the field frequency, $\hat{a}$ and $\hat{a}^\dagger$, respectively, are the annihilation and creation operators for the mode of the cavity field satisfying $[\hat{a}, \hat{a}^\dagger] = 1$. In the Hamiltonian (1), we denote by $\varkappa$ the dispersive part of the third-order nonlinearity of the Kerr-like medium, s_1 and s_2 are the parameters describing the intensity-dependent Stark shifts of the two levels that are due to the virtual transitions to the intermediate relay level when $k = 2$ (i.e. the two-photon process). Also, $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ are the atomic peseudo-spin operators. We define the detuning parameter $\Delta_1 = \omega_A - k\omega_F$, where k is the photon multiplicity and ω_A is the transition frequencies between the levels. We consider the initial state of the atom to be in a coherent superposition state of the excited state $|e\rangle$ and ground state $|g\rangle$, namely

$$|\vartheta, \varphi\rangle = \cos(\vartheta/2)|e\rangle + \sin(\vartheta/2)\exp(-\mathrm{i}\varphi)|g\rangle \quad (2)$$

where φ is the relative phase of the two atomic levels and ϑ denotes the polarization direction. When $\vartheta \to 0$, the excited state is given but when $\vartheta \to \pi$, the wavefunction describes the ground state of the atom.

Also, we assume that the field is initially in the coherent state,

$$|\alpha\rangle = \sum_{n=0}^{\infty} b_n|n\rangle = \sum_{n=0}^{\infty} \exp\left(-\frac{\bar{n}}{2}\right)\frac{\alpha^n}{\sqrt{n!}}|n\rangle \quad (3)$$

where b_n describes the amplitude of state $|n\rangle$ of the mode, $\alpha = \sqrt{(\bar{n})}\exp(\mathrm{i}\eta)$. $\bar{n}$ and η represent the initial average photon number and the phase of the mode, respectively. At time $t = 0$ the field–atom system is in a pure state, thus the initial density operator of the system is given by $\hat{\rho}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_F(0)$, where $\hat{\rho}_F(0) = |\alpha\rangle\langle\alpha|$ and $\hat{\rho}_A(0) = |\vartheta, \varphi\rangle\langle\vartheta, \varphi|$. $\hat{\rho}(0)$ describes the initial value for the field–atom density operator. At any time $t > 0$ the solution of the Schrödinger equation under the Hamiltonian (1) can be written as [1, 15, 16, 18, 19]

$$|\Psi_{\mathrm{AF}}(t)\rangle = \sum_{n=0}^{\infty}\{\Psi_e(n,t)|n,e\rangle + \Psi_g(n,t)|n,g\rangle\} \quad (4)$$

where the coefficient $\Psi_e(n,t)$ and $\Psi_g(n,t)$ are given by

$$\Psi_e(n,t) = \mathrm{e}^{-\mathrm{i}\lambda t R_n}\left\{b_n\left(\cos\Upsilon_n - \mathrm{i}\frac{\delta_n}{\Omega_n}\sin\Upsilon_n\right)\cos(\vartheta/2) - \frac{\mathrm{i}v_n b_{n+k}}{\Omega_n}\sin\Upsilon_n\sin(\vartheta/2)\,\mathrm{e}^{-\mathrm{i}\varphi}\right\} \tag{5}$$

$$\Psi_g(n,t) = \mathrm{e}^{-\mathrm{i}\lambda t R_{n-k}}\left\{b_n\left(\cos\Upsilon_{n-k} + \mathrm{i}\frac{\delta_{n-k}}{\Omega_{n-k}}\sin\Upsilon_{n-k}\right)\sin(\vartheta/2)\,\mathrm{e}^{-\mathrm{i}\varphi} - \frac{\mathrm{i}v_{n-k}b_{n-k}}{\Omega_{n-k}}\sin\Upsilon_{n-k}\cos(\vartheta/2)\right\} \tag{6}$$

where

$$\begin{aligned} R_n &= \frac{\chi}{4}[(2n+k-1)^2 + k^2 - 1] + \frac{1}{2}[n\beta_2 + (n+k)\beta_1]\delta(k-2) \\ \delta_n &= \frac{\Delta}{2} - \chi k[2n+k-1] + \frac{1}{2}[n\beta_2 - (n+k)\beta_1]\delta(k-2) \\ v_n &= \sqrt{\frac{(n+k)!}{n!}} \\ \Omega_n &= \sqrt{\delta_n^2 + v_n^2} \\ \Upsilon_n &= \lambda t\Omega_n \end{aligned} \tag{7}$$

$\Delta = \Delta_1/\lambda$, $\chi = \varkappa/\lambda$ and $\beta_i = s_i/\lambda$, $i = 1, 2$. Here $\lambda\Omega_n$ is the generalized Rabi frequency which depends on the detuning parameter, Stark shift parameters and medium nonlinearity. Equation (4) represents the state of the field system and atom–field system at any time during the evolution. It is to be noted that this result (solution (4)–(7)) generalizes earlier results [20–23]; also it should be emphasized that this result is a special case of the solution given in [1].

With the wavefunction $|\Psi_{\mathrm{AF}}(t)\rangle$, any property related to the atom and the field can be calculated. The reduced density matrix of the field–atom system can be written as

$$\begin{aligned} \rho_F(t) &= \mathrm{tr}_A[|\Psi_{\mathrm{AF}}(t)\rangle\langle\Psi_{\mathrm{AF}}(t)|] \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{\Psi_e(n,t)\Psi_e^*(m,t)|n\rangle\langle m| + \Psi_g(n,t)\Psi_g^*(m,t)|n\rangle\langle m|\} \\ &= |C\rangle\langle C| + |S\rangle\langle S| \end{aligned} \tag{8}$$

where

$$|C\rangle = \sum_{n=0}^{\infty}\Psi_e(n,t)|n\rangle \qquad |S\rangle = \sum_{n=0}^{\infty}\Psi_g(n,t)|n\rangle \tag{9}$$

$$\langle n|\rho_F(t)|m\rangle = \Psi_e(n,t)\Psi_e^*(m,t) + \Psi_g(n,t)\Psi_g^*(m,t). \tag{10}$$

Also, the reduced density matrix of the atom can be written as

$$\begin{aligned} \rho_A(t) &= \mathrm{tr}_F[|\Psi_{\mathrm{AF}}(t)\rangle\langle\Psi_{\mathrm{AF}}(t)|] \\ &= \sum_{n=0}^{\infty}\{\Psi_e(n,t)\Psi_e^*(n,t)|e\rangle\langle e| + \Psi_g(n,t)\Psi_g^*(n,t)|g\rangle\langle g| \\ &\quad + \Psi_e(n,t)\Psi_g^*(n,t)|e\rangle\langle g| + \Psi_e^*(n,t)\Psi_g(n,t)|g\rangle\langle e|\}. \end{aligned} \tag{11}$$

Then, we can calculate the expectation values for any function $F(\hat{a},\hat{a}^\dagger)$ in the usual manner:

$$\langle F\rangle = \mathrm{tr}_F[\rho_F(t)F(\hat{a},\hat{a}^\dagger)] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\langle n|\rho_F(t)|m\rangle\langle m|F(\hat{a},\hat{a}^\dagger)|n\rangle \tag{12}$$

while for any atomic variable G we use

$$\langle G\rangle = \mathrm{tr}_A[\rho_A(t)G]. \tag{13}$$

Once these quantities have been calculated, we can discuss many features concerning the field and the atom. By using the reduced density operators $\rho_A(t)$ given by equation (11), we can evaluate the expectation values of the atomic variables $\langle\sigma_x(t)\rangle$, $\langle\sigma_y(t)\rangle$ and $\langle\sigma_z(t)\rangle$, and we arrive at the following expressions:

$$\langle\sigma_x(t)\rangle = \mathrm{tr}_A[\rho_A(t)\sigma_x] = \sum_{n=0}^{\infty}\{U_R\cos[\lambda t(R_n - R_{n-k})] + U_V\sin[\lambda t(R_n - R_{n-k})]\} \tag{14}$$

$$\langle\sigma_y(t)\rangle = \mathrm{tr}_A[\rho_A(t)\sigma_y] = \sum_{n=0}^{\infty}\{U_R\sin[\lambda t(R_n - R_{n-k})] - U_V\cos[\lambda t(R_n - R_{n-k})]\} \tag{15}$$

$$\begin{aligned}U_R = \frac{\sin(\vartheta)}{2}&\left([X_n\cos\varphi + Y_n\sin\varphi]|b_n|^2 + \frac{\upsilon_n\upsilon_{n-k}}{\Omega_n\Omega_{n-k}}|b_{n-k}b_{n+k}|\sin\Upsilon_n\sin\Upsilon_{n-k}\cos(\varphi - k\eta)\right)\\&- \frac{\upsilon_{n-k}}{\Omega_{n-k}}|b_nb_{n-k}|\sin\Upsilon_{n-k}\cos^2(\vartheta/2)\left(\cos\Upsilon_n\sin k\eta - \frac{\delta_n}{\Omega_n}\sin\Upsilon_n\cos k\eta\right)\\&+ \frac{\upsilon_n}{\Omega_n}|b_nb_{n+k}|\sin\Upsilon_n\sin^2(\vartheta/2)\left(\cos\Upsilon_n\sin k\eta - \frac{\delta_{n-k}}{\Omega_{n-k}}\sin\Upsilon_{n-k}\cos k\eta\right)\end{aligned}$$

$$\begin{aligned}U_V = \frac{\sin(\vartheta)}{2}&\left([X_n\sin\varphi - Y_n\cos\varphi]|b_n|^2 - \frac{\upsilon_n\upsilon_{n-k}}{\Omega_n\Omega_{n-k}}|b_{n-k}b_{n+k}|\sin\Upsilon_n\sin\Upsilon_{n-k}\sin(\varphi - k\eta)\right)\\&+ \frac{\upsilon_{n-k}}{\Omega_{n-k}}|b_nb_{n-k}|\sin\Upsilon_{n-k}\cos^2(\vartheta/2)\left(\cos\Upsilon_n\cos k\eta + \frac{\delta_n}{\Omega_n}\sin\Upsilon_n\sin k\eta\right)\\&- \frac{\upsilon_n}{\Omega_n}|b_nb_{n+k}|\sin\Upsilon_n\sin^2(\vartheta/2)\left(\cos\Upsilon_{n-k}\cos k\eta + \frac{\delta_{n-k}}{\Omega_{n-k}}\sin\Upsilon_{n-k}\sin k\eta\right)\end{aligned}$$

$$\begin{aligned}\langle\sigma_z(t)\rangle = \mathrm{tr}_A[\rho_A(t)\sigma_z] = \frac{1}{2}\sum_{n=0}^{\infty}|b_n|^2&\left\{\left(\cos^2\Upsilon_n + \frac{\delta_n^2}{\Omega_n^2}\sin^2\Upsilon_n\right)\cos^2(\vartheta/2)\right.\\&\left.- \left(\cos^2\Upsilon_{n-k} + \frac{\delta_{n-k}^2}{\Omega_{n-k}^2}\sin^2\Upsilon_{n-k}\right)\sin^2(\vartheta/2)\right\}\\&+ \sum_{n=0}^{\infty}\left\{|b_{n+k}|^2\left(\frac{\upsilon_n}{2\Omega_n}\sin\Upsilon_n\sin(\vartheta/2)\right)^2\right.\\&\left.- |b_{n-k}|^2\left(\frac{\upsilon_{n-k}}{2\Omega_{n-k}}\sin\Upsilon_{n-k}\cos(\vartheta/2)\right)^2\right\}\\&- \sin(\vartheta)\sum_{n=0}^{\infty}\left\{\left(\frac{\upsilon_n}{4\Omega_n}|b_nb_{n+k}|\sin 2\Upsilon_n\right.\right.\\&\left.- \frac{\upsilon_{n-k}}{4\Omega_{n-k}}|b_nb_{n-k}|\sin 2\Upsilon_{n-k}\right)\sin(\varphi - k\eta)\\&\left.- \left(\frac{\upsilon_n\delta_n}{2\Omega_n^2}|b_nb_{n+k}|\sin^2\Upsilon_n - \frac{\upsilon_{n-k}\delta_{n-k}}{2\Omega_{n-k}^2}|b_nb_{n-k}|\sin^2\Upsilon_{n-k}\right)\cos(\varphi - k\eta)\right\}\end{aligned} \tag{16}$$

$$\left.\begin{aligned} X_n &= \cos\Upsilon_n \cos\Upsilon_{n-k} - \frac{\delta_n\delta_{n-k}}{\Omega_n\Omega_{n-k}} \sin\Upsilon_n \sin\Upsilon_{n-k} \\ Y_n &= \frac{\delta_n}{\Omega_n} \sin\Upsilon_n \cos\Upsilon_{n-k} + \frac{\delta_{n-k}}{\Omega_{n-k}} \sin\Upsilon_{n-k} \cos\Upsilon_n . \end{aligned}\right\} \tag{17}$$

By using the expectation values of the atomic variables, we can calculate the atomic Wehrl density and atomic Wehrl entropy of a two-level atom in the presence of the Kerr medium and Stark shift; this will be the subject of the following sections.

3. Atomic Q-function, atomic Wehrl-density $S_q(t)$ and atomic Wehrl entropy $S_{\mathrm{AW}}(t)$

We use the Q-function which will be the basis for calculating the atomic Wehrl entropy. This quasiprobability distribution is defined as [22, 23]

$$Q_A(\Theta, \Phi, t) = \frac{1}{\pi} \langle \Theta, \Phi | \rho_A(t) | \Theta, \Phi \rangle \tag{18}$$

where $\rho_A(t)$ is the reduced density of the atom given by equation (11) and $\rho_A(0) = |\vartheta, \varphi\rangle\langle\vartheta, \varphi|$, and $|\vartheta, \varphi\rangle$ represents the atom initially in the superposition state which it defined in equation (2), with a similar definition for the atomic coherent state $|\Theta, \Phi\rangle$, namely

$$|\Theta, \Phi\rangle = \cos(\Theta/2)|e\rangle + \sin(\Theta/2)\, \mathrm{e}^{\mathrm{i}\Phi}|g\rangle \tag{19}$$

where Θ and Φ are the atomic phase space parameters. Then we recast equation (18) in the following form:

$$Q_A(\Theta, \Phi, t) = \frac{1}{\pi} \left\{ \frac{1}{2} + \langle \sigma_z(t) \rangle \cos\Theta + [\langle \sigma_x(t) \rangle \cos\Phi + \langle \sigma_y(t) \rangle \sin\Phi] \sin\Theta \right\} \tag{20}$$

where the expectation values of the atomic variables $\langle \sigma_x(t) \rangle$, $\langle \sigma_y(t) \rangle$ and the atomic inversion $\langle \sigma_z(t) \rangle$ are given by equations (14),(15) and (16).

In a parallel definition for the field Wehrl entropy [10–12] we define the atomic Wehrl entropy

$$S_{\mathrm{AW}}(t) = \int_0^{2\pi} \int_0^{\pi} S_q(\Theta, \Phi, t) \sin\Theta \, \mathrm{d}\Theta \, \mathrm{d}\Phi \tag{21}$$

where $S_q(\Theta, \Phi, t)$ is the atomic Wehrl density (see [13]), which is given by

$$S_q(\Theta, \Phi, t) = -Q_A(\Theta, \Phi, t) \ln Q_A(\Theta, \Phi, t). \tag{22}$$

The atomic Wehrl density and atomic Wehrl entropy are very important measurements in the field of quantum information and quantum computation [13].

We see from equation (22) that the atomic Wehrl density is dependent on the atomic phase space parameters Θ and Φ. Also, one can see that the atomic Wehrl entropy is a Shannon entropy for the atomic Q-function; then it can be defined as Shannon Wehrl entropy. Also, we see from equation (20) if $\Theta = 0$ then $Q_A(0, \Phi, t)$ is Φ independent and dependent only on the atomic variable $\langle \sigma_z(t) \rangle$ through the following relation:

$$Q_A(0, 0, t) = \frac{1}{\pi} \left\{ \frac{1}{2} + \langle \sigma_z(t) \rangle \right\}. \tag{23}$$

Then the atomic Wehrl density in this case is connected with the expectation value of the population inversion by the following relation:

$$S_q(0, 0, t) = -\frac{1}{\pi} \left(\frac{1}{2} + \langle \sigma_z(t) \rangle \right) \ln \left\{ \frac{1}{\pi} \left(\frac{1}{2} + \langle \sigma_z(t) \rangle \right) \right\}. \tag{24}$$

By integrating the atomic Wehrl density for the variables Θ and Φ, we can write the marginal distributions of the atomic Wehrl density as follows:

$$S_q(\Phi) = \int_0^{\pi} S_q(\Theta, \Phi, t) \sin\Theta \, \mathrm{d}\Theta \tag{25}$$

$$S_q(\Theta) = \int_0^{2\pi} S_q(\Theta, \Phi, t) \, \mathrm{d}\Phi. \tag{26}$$

In what follows, we try to examine the influences of the Kerr nonlinearity parameter, detuning and Stark shift on the marginal distribution $S_q(\Phi)$.

4. Numerical results and discussion

We present the numerical results of the atomic Wehrl entropy $S_q(t)$ of (22) by using the definition of the atomic Q_A-function. We examine the roles played by the detuning, Kerr medium and Stark shift. In figures 1–7, we have studied the temporal behaviour of the atomic Wehrl density of the field in one- and two-photon Jaynes Cummings models, but marginal distribution of the atomic Wehrl density of (25) is discussed in figure 8. A comparison of the atomic Wehrl entropy $S_{\mathrm{AW}}(t)$ of (21) and von Neumann entropy $S_A(t) = -\mathrm{Tr}\, \rho_A(t) \ln \rho_A(t)$ is presented in figure 9. We recall that time t has been scaled; one unit of time is given by the inverse of the coupling constant λ. One can see that the Wehrl entropy does not take negative values, which follows from the fact that $0 \leqslant Q_A(\Theta, \Phi, t) \leqslant 1/\pi$ and the $Q_A(\Theta, \Phi, t)$ function can never be so concentrated as to make $S_{\mathrm{AW}}(t)$ negative. So for the maximum value of the atomic Wehrl density $S_q = \ln(\pi)/\pi$, we have plotted the time evolution of the atomic Wehrl density $S_q(\Theta, \Phi, t)$ against the scaled time for $0 \leqslant \lambda t \leqslant 100$, for $k = 1$ (one-photon process) and $k = 2$ (two-photon process). We restrict our discussion when the atom is initially in the superposition sate, so we set $\vartheta = \pi/2$ and $\varphi = \eta = \pi/4$.

4.1. Effect of the mean photon number

Figure 1 shows the influence of the mean photon number $\bar{n}$ on the atomic Wehrl density and in the absence of both Kerr medium and Stark shift. It is to be noted that the atomic Wehrl density evolves to a minimum value at the revival time $t_R = 2\pi\sqrt{\bar{n}} = (4\pi, 8\pi, 12\pi)$, see figures 1(*a*)–(*c*). To see the atomic Wehrl density influenced by increasing the mean photon number, we set $\bar{n} = 25$ in figure 1(*b*) and $\bar{n} = 36$ in figure 1(*c*). We see that the minimum value at the revival time and the amplitudes of the atomic Wehrl density as the mean photon number $\bar{n}$ increase. Also, the more the mean photon number $\bar{n}$ increased the more the oscillations at the revival time increased.

The two-photon process ($k = 2$) is considerd in figure 2. We show the effect of the mean photon number $\bar{n}$ on the atomic Wehrl density $S_q(t)$. It is observed that increasing the mean photon number leads to the increase in the minimum value of the atomic Wehrl density. Also the atomic Wehrl density $S_q(t)$ is a periodic function and it takes a maximum value at the revival time $t_R = \frac{n\pi}{\lambda}$ ($n = 1, 2, \ldots$), and a minimum value at the half of revival time $t_R/2$. It is clear that there is a great difference between the one-photon process $k = 1$ and two-photon process $k = 2$, see figure 1.

4.2. Effect of the detuning parameter

Figure 3 shows the influence of the detuning parameter on the atomic Wehrl density. We set different values of the detuning parameter ($\Delta = 5, 10, 20$), $\bar{n} = 25$. When $\Delta = 5$, the

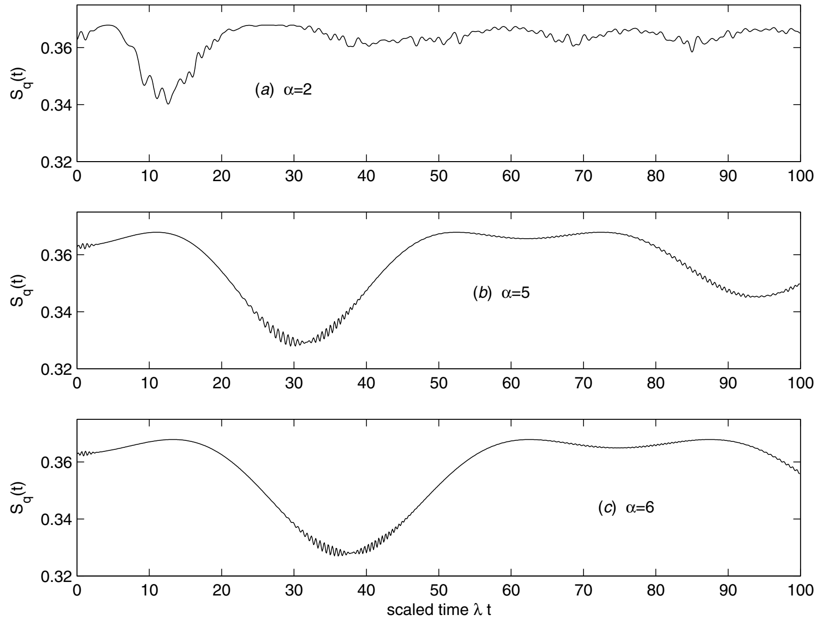

Figure 1. The time evolution of the atomic Wehrl density $S_q(t)$ of a two-level atom interacting with a single mode, the atomic phase space parameters being $\Theta = \pi/2$ and $\Phi = \pi/4$. The atom is initially in the superposition state $\vartheta = \pi/2, \varphi = \eta = \pi/4$, for the parameters $k = 1$, $\Delta = \chi = 0$ and with different values of $\bar{n} = |\alpha|^2$.

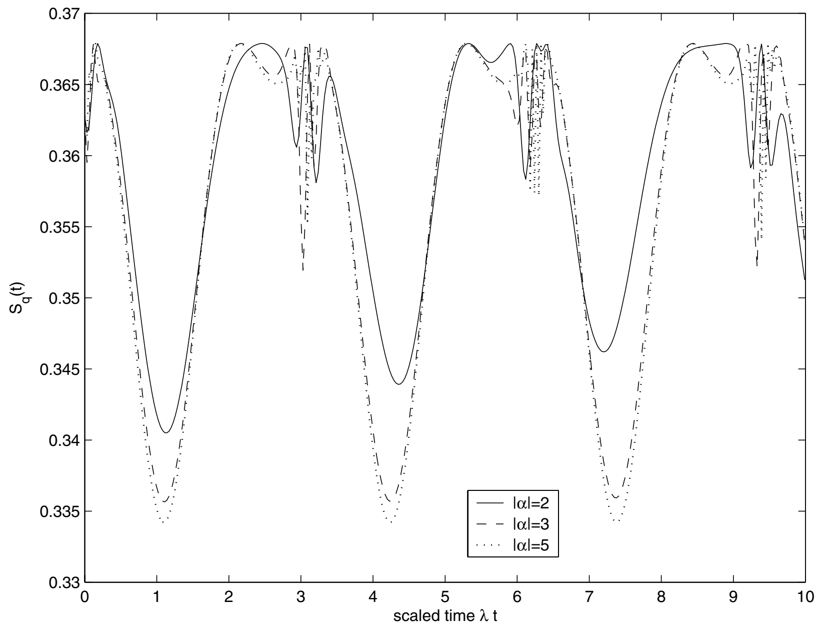

Figure 2. The time evolution of the atomic Wehrl density $S_q(t)$ of a two-level atom interacting with a single mode, the atomic phase space parameters being $\Theta = \pi/2$ and $\Phi = \pi/4$. The atom is initially in the superposition state $\vartheta = \pi/2, \varphi = \eta = \pi/4$, with different values of the mean photon number $\bar{n} = |\alpha|^2$, $k = 2$, in the case of neglecting the Stark effect and Kerr medium parameters $\chi = \Delta = 0$.

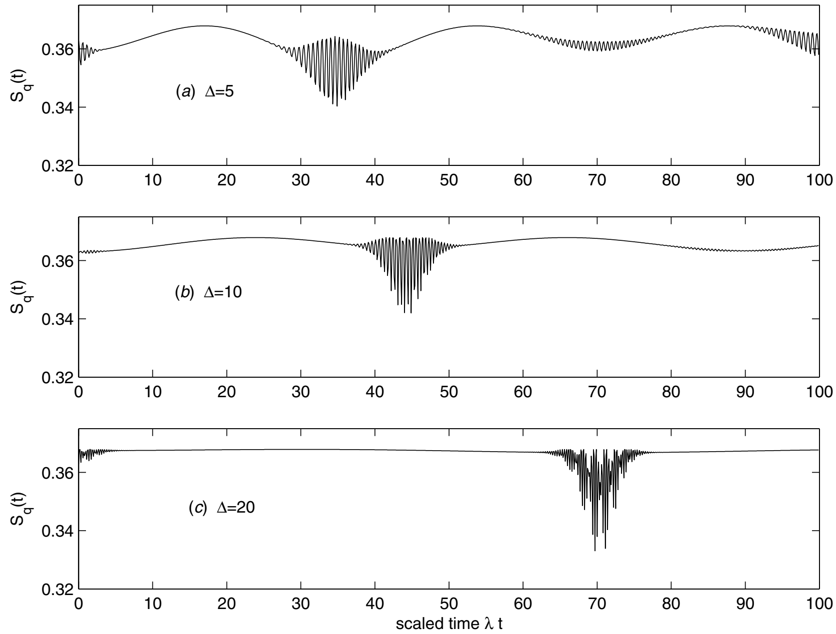

Figure 3. The time evolution of the atomic Wehrl density $S_q(t)$ of a two-level atom interacting with a single mode, the atomic phase space parameters being $\Theta = \pi/2$ and $\Phi = \pi/4$. The atom is initially in the superposition state $\vartheta = \pi/2, \varphi = \eta = \pi/4$, for the parameters $\bar{n} = 25, k = 1, \chi = 0$ and with different values of Δ.

atomic Wehrl density $S_q(t)$ takes a minimum value at the revival time $t_R = 2\pi\sqrt{\bar{n} + \frac{\Delta^2}{4}}$, and a maximum value at the half of revival time $t_R/2$. As the detuning parameter is increased the minimum value of the atomic Wehrl density is also increased. Also, the rapid oscillations around the minimum value are decreased. In this case super oscillations are noted on increasing the detuning parameter (see figures 3(*b*) and (*c*)).

The influence of the detuning parameter on the atomic Wehrl density in the case of two-photon process is examined in figure 4. The influence of the detuning parameter is considered through the parameter Δ. We set three values of the detuning parameter $\Delta = 5, 10, 20$, the mean photon number is taken to be $\bar{n} = 25$ and all the other parameters are the same as in figure 2. We note that the minimum value of the atomic Wehrl entropy $S_q(t)$ decreases as the detuning parameter is increased.

4.3. Effect of the Kerr medium

To visualize the influence of the Kerr-like medium for the one-photon process ($k = 1$) on the atomic Wehrl density, we considered the influence of the Kerr medium through the parameter χ. We set different values of the Kerr-like medium parameter χ as follows: (*a*) $\chi = 0.05$, (*b*) $\chi = 0.1$, (*c*) $\chi = 0.5$. It is clear that the minimum values for the system with the Kerr-like medium effect are higher than those without the Kerr effect. Also regularity is noted and explored (see figure 5).

In order to see how the atomic Wehrl density influenced by the nonlinear medium through the parameter χ, we set $\chi = 0.05$ in figure 6(*a*) and $\chi = 0.1, 0.5$, in figures 6(*b*) and (*c*) for the two-photon process ($k = 2$). As the time increases the atomic Wehrl density

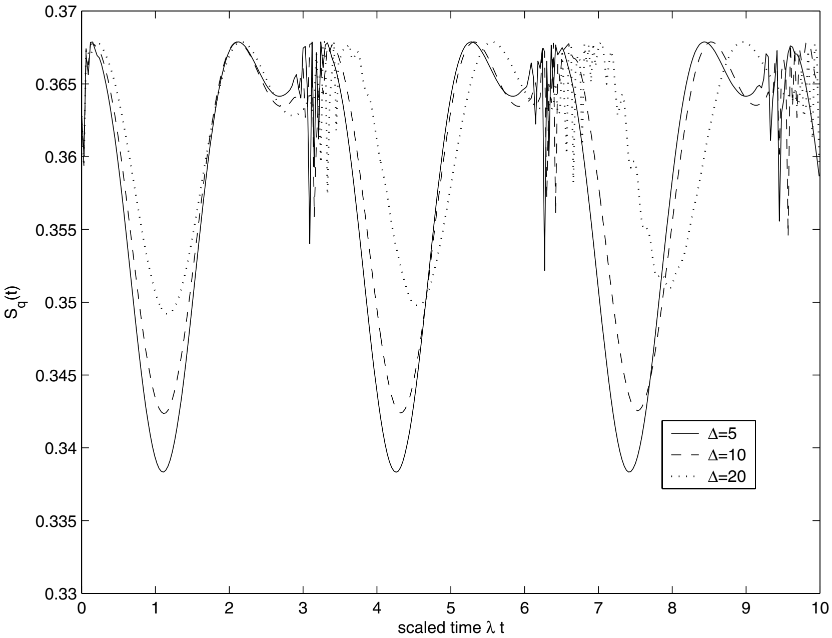

Figure 4. The time evolution of the atomic Wehrl density $S_q(t)$ of a two-level atom interacting with a single mode, the atomic phase space parameters being $\Theta = \pi/2$ and $\Phi = \pi/4$. The atom is initially in the superposition state $\vartheta = \pi/2, \varphi = \eta = \pi/4$, for the parameters $\bar{n} = 25, k = 2$, $\chi = 0$ in the case of neglecting the Stark effect and with different values of Δ.

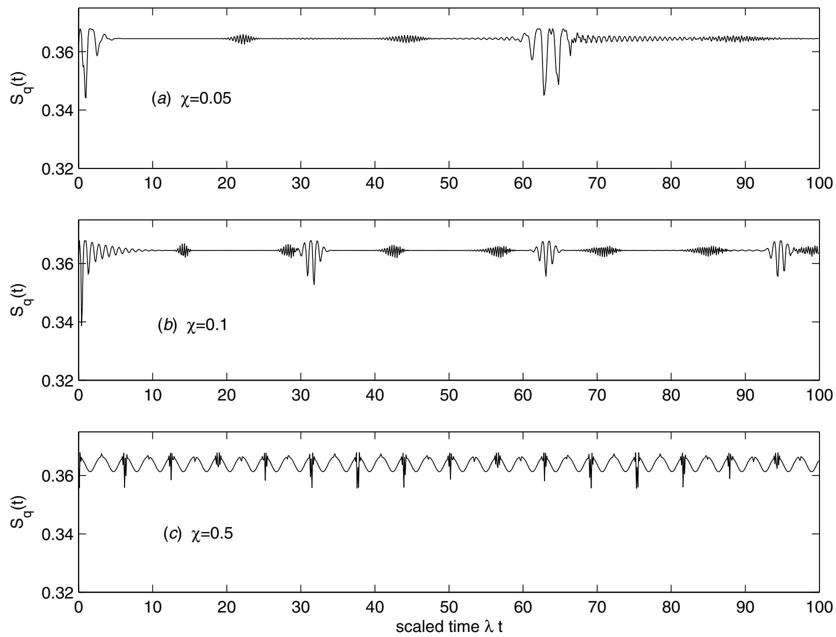

Figure 5. The time evolution of the atomic Wehrl density $S_q(t)$ of a two-level atom interacting with a single mode, the atomic phase space parameters being $\Theta = \pi/2$ and $\Phi = \pi/4$. The atom is initially in the superposition state $\vartheta = \pi/2, \varphi = \eta = \pi/4$, for the parameters $\bar{n} = 25, k = 1, \Delta = 0$ and with different values of χ.

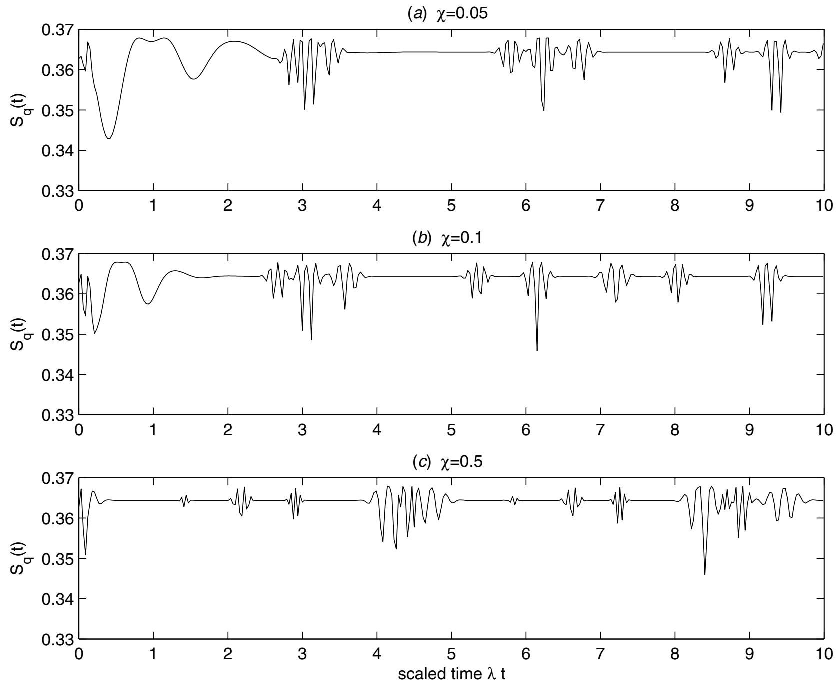

Figure 6. The time evolution of the atomic Wehrl density $S_q(t)$ of a two-level atom interacting with a single mode, the atomic phase space parameters being $\Theta = \pi/2$ and $\Phi = \pi/4$. The atom is initially in the superposition state $\vartheta = \pi/2$, $\varphi = \eta = \pi/4$, for the parameters $\bar{n} = 25$, $k = 2$, $\Delta = 0$ in the case of neglecting the Stark effect and with different values of χ.

returns to a maximum value at the first stage of the time evolution, see figure 6(*a*). As the Kerr-like medium increases we get more oscillations at the period of time. When $\chi = 0.5$, the maximum value of the atomic Wehrl density increases and sharp peaks are observed with some kind of periodicity with more oscillation at the same period of time

The Kerr effects on the atomic Wehrl density have been shown to give the long surviving of the entanglement between the atom and the field, and the amplitude becomes much smaller, which means that the Kerr-like medium can be used in the laboratory to generate maximum entangled states. In our calculations the surprising result is that with increasing the Kerr-like medium, the oscillations become smaller and closer to the maximum value with long surviving of the entanglement. This result is quite interesting, since the recent experimental observation of the entanglement stated that the principal aim of generating maximum entangled state is to have the long surviving of the entanglement (see figure 6).

4.4. Effect of the Stark shift

As is visible from figures 7(*a*)–(*c*), the effects of the dynamic Stark shift for $k = 2$ are more pronounced when $\beta_1 = 1/\beta_2$, and when the atom is initially in the superposition state. As β_1 is increased the minimum values of the atomic Wehrl density increase but the maximum value is independent of the values of the Stark shift parameter β_1. Then the presence of the Stark shift leads to disentanglement between the atom and the field. Hence, one can see that the influence of the Stark shift on the von Neumann entropy is similar to the atomic Wehrl density (see figures 7(*d*)–(*f*)).

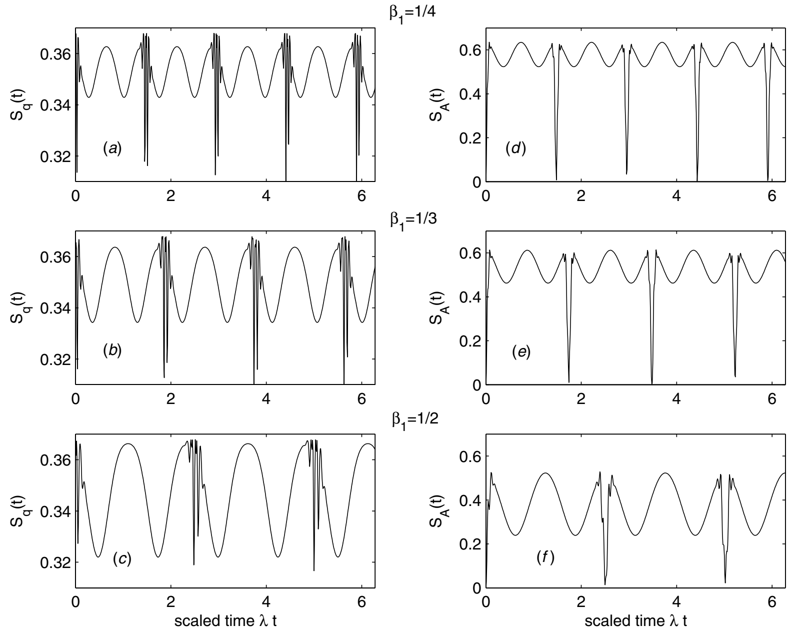

Figure 7. The time evolution of the atomic Wehrl density $S_q(t)$ ((*a*), (*b*) and (*c*)) and von Neumann entropy $S_A(t)$ ((*d*), (*e*) and (*f*)) of a two-level atom interacting with a single-mode, the atomic phase space parameters being $\Theta = \pi/2$ and $\Phi = \pi/4$. The atom is initially in the superposition state $\vartheta = \pi/2, \varphi = \eta = \pi/4$, for the parameters $\bar{n} = 25$, $k = 2$, $\Delta = \chi = 0$ and with different values of β_1 and $\beta_2 = 1/\beta_1$.

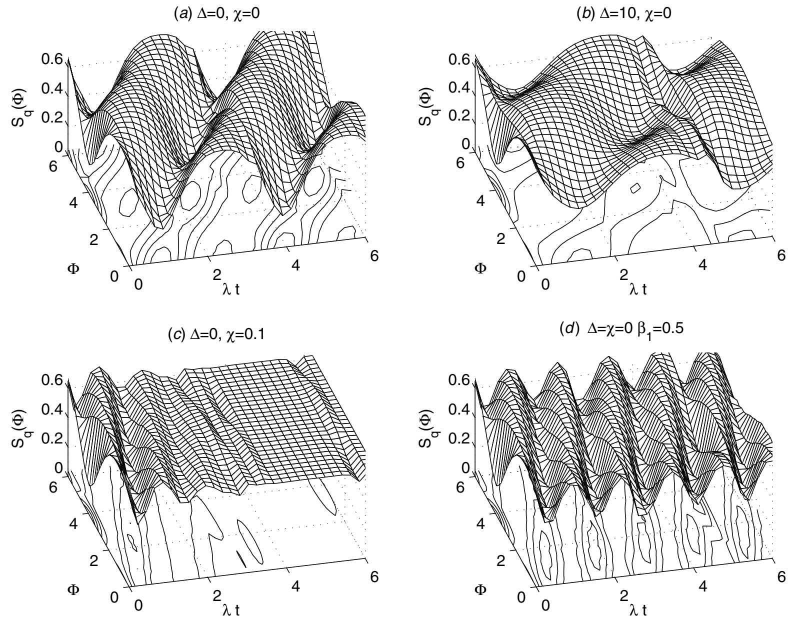

Figure 8. The marginal distribution of the atomic Wehrl density $S_q(\Phi)$ against the scaled time λt and Φ. The atom is initially in the superposition state $\vartheta = \pi/2, \varphi = \eta = \pi/4$, for the parameters $\bar{n} = 25$, $k = 2$ and for different values of χ, Δ and β_1.

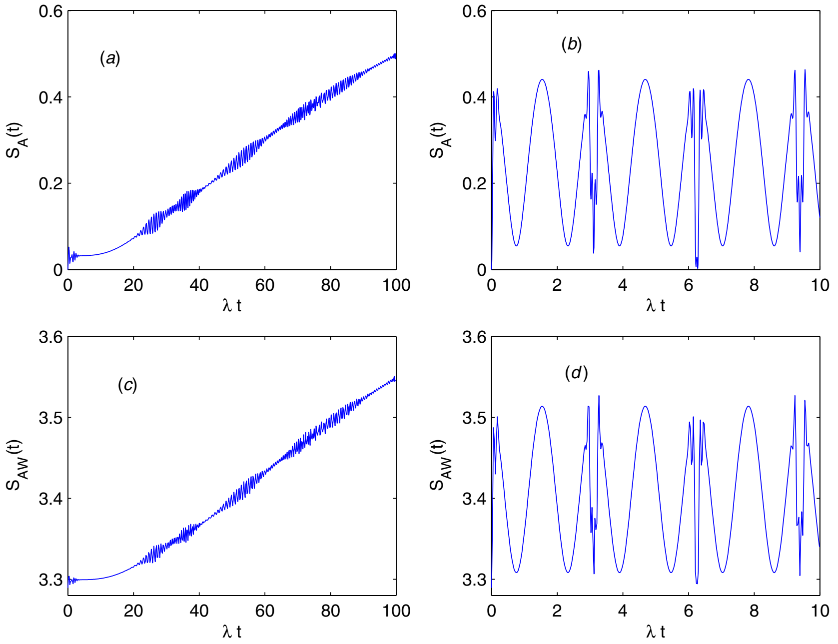

Figure 9. The evolution of the von Neumann entropy $S_A(t)$ ((*a*) and (*b*)), atomic Wehrl entropy $S_{\mathrm{AW}}(t)$ ((*c*) and (*d*)) as a function of the scaled time λt. The atom is initially in the superposition state $\vartheta = \pi/2$, $\varphi = \eta = \pi/4$, for the parameters $\bar{n} = 25$, $\Delta = \chi = 0$ in the case of neglecting the Stark shift for $k = 2$.

4.5. Marginal distribution $S_q(\Phi)$

A comparison between the Kerr-like medium, Stark shift and the detuning is plotted in figure 8. It should be noted that the detuning has the usual effect by elongating the revival time. On the other hand, the Kerr effect adds small oscillations at the earlier interaction time and then $S_q(\Phi)$ takes constant values close to the maximum value. While, the regular behaviour with some kind of periodicity is achieved when Stark shift takes place (see figure 8).

Here it is interesting to mention a recent paper [24] in which the authors proposed to quantify the entanglement of pure states of $N \times N$ bipartite quantum systems by defining its Husimi distribution.

In figure 9, we see that the atomic Wehrl entropy increases with the time in the one-photon case while the periodic behaviour has been observed in the two-photon case in *agreement with the behaviour of the von Neumann entropy*.

5. Conclusion

We have investigated the evolution of the atomic Wehrl entropy in multi-quanta transition in the presence of the Kerr-like medium and Stark shift. We examined different effects on the dynamics of the atomic Wehrl entropy and its density. It has been observed that the atomic Wehrl entropy gives equivalent results to the von Neumann entropy for all the involved parameters of our system, so that we focus our attention on the atomic Wehrl density. In general, we found that the Kerr medium influences the atomic Wehrl density significantly.

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